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An integrable semi-discretization of the Camassa–Holm equation and its determinant solution

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Abstract

An integrable semi-discretization of the Camassa–Holm (CH) equation is presented. The keys of its construction are bilinear forms and determinant structure of solutions of the CH equation. Determinant formulas of *N*-soliton solutions of the continuous and semi-discrete Camassa–Holm equations are presented. Based on determinant formulas, we can generate multi-soliton, multi-cuspon and multi-soliton–cuspon solutions. Numerical computations using the integrable semi-discrete Camassa–Holm equation are performed. It is shown that the integrable semi-discrete Camassa–Holm equation gives very accurate numerical results even in the cases of cuspon–cuspon and soliton–cuspon interactions. The numerical computation for an initial value condition, which is not an exact solution, is also presented.

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1. Introduction

The Camassa–Holm (CH) equation,

$$w_T + 2\kappa^2 w_X - w_{TXX} + 3ww_X = 2w_X w_{XX} + ww_{XXX}, \tag{1.1}$$

has attracted considerable interest since it has been derived as a model equation for shallow-water waves [1]. Here, $w = w(X, T)$, κ is a positive parameter and the subscripts T and X appended to $w(X, T)$ denote partial differentiation. Originally, this equation has been found in a mathematical search of recursion operators connected with integrable partial differential equations [2]. The CH equation has been shown to be completely integrable. In the case of $\kappa = 0$, the CH equation admits peakon solutions which are represented by piecewise analytic functions [3]. Schiff obtained 1- and 2-soliton solutions in a parametric form by using the Bäcklund transformation [4]. An approach based on the inverse scattering transform method (IST) provides an explicit form of the inverse mapping in terms of Wronskian [5–8]. The

N-soliton solution was also constructed by using the Hirota bilinear method [9–12] (see also [13]). The key point of computations of the *N*-soliton solution by the Hirota bilinear method in those papers is the relationship between the CH equation and the AKNS shallow-water wave equation [14]. When $\kappa \neq 0$, cusped solitary wave solutions, as well as analytic soliton solutions, were found in [15–17]. In [18, 19], the interaction of cusped soliton (cuspon) was studied in detail.

It is extremely difficult to perform numerical computations of the CH equation due to the singularities of cuspon and peakon solutions. So far, several numerical computations of the CH equation were presented [20–26]. However, none of these numerical methods gives satisfactory results for soliton–cuspon and cuspon–cuspon interactions.

Integrable discretizations of soliton equations have recently received considerable attention [27, 28]. Ablowitz and Ladik proposed how to construct integrable discrete analogues of soliton equations based on Lax pairs [29, 30]. Hirota proposed another method to construct integrable discrete analogues of soliton equations based on bilinear equations [31–33]. Applications of integrable discretizations of soliton equations were considered in various fields [34–36].

The purpose of this paper is to present an integrable semi-discretization of the CH equation through a bilinear approach and to perform numerical computations by using integrable scheme. We show that the integrable scheme gives very accurate numerical results even for the soliton–soliton and cuspon–cuspon interactions.

The outline of this paper is as follows. In section 2, we give bilinear forms and a determinant formula of the *N*-soliton solution of equation (1.1). As far as we are concerned, bilinear forms directly related to the CH equation have not been known yet. These bilinear forms can help us to understand mathematical structure of the CH equation more thoroughly. Based on these bilinear equations, we obtain the determinant formula for the CH equation by using the determinant technique [37–40]. In section 3, we give an integrable semi-discrete Camassa–Holm equation and its determinant formula of the *N*-soliton solution. In section 4, we present results of numerical computations by using the proposed integrable semi-discrete CH equation.

2. Bilinear equations and determinant formulas of the Camassa–Holm equation

In this section, we give bilinear equations and the *N*-soliton solution of the CH equation. All of the existing works in constructing *N*-soliton solutions of the CH equation using the Hirota bilinear method take advantage of the relationship between the CH equation and the AKNS shallow-water wave equation [9–12, 14]. So far, the bilinear forms which are directly related to the CH equation remain unknown. In this paper, we give bilinear equations directly obtained from the CH equation and derive the determinant solution by using the determinant technique. Our formulation in this section is crucial for the derivation of integrable semi-discretization of the CH equation.

Lemma 2.1. *Bilinear equations*

$$\begin{cases} \left(\frac{1}{2}D_tD_x - 1\right) f \cdot f = -gh, \\ \frac{1}{2}D_t(D_y - 2cD_x) f \cdot f = -D_x g \cdot h, \\ \frac{1}{2}D_sD_x f \cdot f = D_t g \cdot h, \\ \left(\frac{1}{2}D_s(D_y - 2cD_x) - 2\right) f \cdot f = (D_tD_x - 2) g \cdot h, \end{cases} \tag{2.1}$$

have a determinant solution

$$f = \tau_0, \qquad g = \tau_1, \qquad h = \tau_{-1},$$

$$\tau_n = \begin{vmatrix} \psi_1^{(n)} & \psi_1^{(n+1)} & \cdots & \psi_1^{(n+N-1)} \\ \psi_2^{(n)} & \psi_2^{(n+1)} & \cdots & \psi_2^{(n+N-1)} \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)} & \psi_N^{(n+1)} & \cdots & \psi_N^{(n+N-1)} \end{vmatrix},$$

where

$$\psi_i^{(n)} = a_{i,1}(p_i - c)^n \, \mathrm{e}^{\xi_i} + a_{i,2}(q_i - c)^n \, \mathrm{e}^{\eta_i},$$

$$\xi_i = p_i x + p_i^2 y + \frac{1}{p_i - c} t + \frac{1}{(p_i - c)^2} s + \xi_{i0},$$

$$\eta_i = q_i x + q_i^2 y + \frac{1}{q_i - c} t + \frac{1}{(q_i - c)^2} s + \eta_{i0}.$$

Proof. Consider the following Casorati determinant solution,

$$\tau_n = \begin{vmatrix} \psi_1^{(n)} & \psi_1^{(n+1)} & \cdots & \psi_1^{(n+N-1)} \\ \psi_2^{(n)} & \psi_2^{(n+1)} & \cdots & \psi_2^{(n+N-1)} \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)} & \psi_N^{(n+1)} & \cdots & \psi_N^{(n+N-1)} \end{vmatrix}, \tag{2.2}$$

where $\psi_i^{(n)}$'s are arbitrary functions of four continuous independent variables, x, y, t and s, which satisfy linear dispersion relations:

$$\partial_x \psi_i^{(n)} = \psi_i^{(n+1)} + c\psi_i^{(n)}, \tag{2.3}$$

$$\partial_y \psi_i^{(n)} = \partial_x^2 \psi_i^{(n)},$$
$$= \psi_i^{(n+2)} + 2c\psi_i^{(n+1)} + c^2\psi_i^{(n)}, \tag{2.4}$$

$$\partial_t \psi_i^{(n)} = \psi_i^{(n-1)}, \tag{2.5}$$

$$\partial_s \psi_i^{(n)} = \psi_i^{(n-2)}. \tag{2.6}$$

Thus we can choose $\psi_i^{(n)}$ as follows:

$$\psi_i^{(n)} = a_{i,1}(p_i - c)^n \, \mathrm{e}^{\xi_i} + a_{i,2}(q_i - c)^n \, \mathrm{e}^{\eta_i},$$
$$\xi_i = p_i x + p_i^2 y + \frac{1}{p_i - c} t + \frac{1}{(p_i - c)^2} s + \xi_{i0},$$
$$\eta_i = q_i x + q_i^2 y + \frac{1}{q_i - c} t + \frac{1}{(q_i - c)^2} s + \eta_{i0}.$$

For simplicity, we introduce a convenient notation,

$$|n_1, n_2, \ldots, n_N| = \begin{vmatrix} \psi_1^{(n_1)} & \psi_1^{(n_2)} & \cdots & \psi_1^{(n_N)} \\ \psi_2^{(n_1)} & \psi_2^{(n_2)} & \cdots & \psi_2^{(n_N)} \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n_1)} & \psi_N^{(n_2)} & \cdots & \psi_N^{(n_N)} \end{vmatrix}. \tag{2.7}$$

In this notation, the solution for the above bilinear forms, τ_n, is rewritten as

$$\tau_n = |n, n+1, \ldots, n+N-1|. \tag{2.8}$$

We show that the above τ_n actually satisfies the bilinear equations (2.1) by using the Laplace expansion technique [39, 40].

The differential formulas for τ are given by

$$(\partial_x - Nc)\tau_n = |n, n+1, \ldots, n+N-2, n+N|, \tag{2.9}$$

$$\begin{aligned}(\partial_y - 2c\partial_x + Nc^2)\tau_n = {}& |n, n+1, \ldots, n+N-2, n+N+1| \\ & - |n, n+1, \ldots, n+N-3, n+N-1, n+N|,\end{aligned} \tag{2.10}$$

$$\partial_t\tau_n = |n-1, n+1, \ldots, n+N-1|, \tag{2.11}$$

$$\partial_s\tau_n = |n-2, n+1, \ldots, n+N-1| - |n-1, n, n+2, \ldots, n+N-1|, \tag{2.12}$$

$$(\partial_t(\partial_x - Nc) - 1)\tau_n = |n-1, n+1, \ldots, n+N-2, n+N|, \tag{2.13}$$

$$\begin{aligned}\partial_t(\partial_y - 2c\partial_x + Nc^2)\tau_n = {}& |n-1, n+1, \ldots, n+N-2, n+N+1| \\ & - |n-1, n+1, \ldots, n+N-3, n+N-1, n+N|,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\partial_s(\partial_x - Nc)\tau_n = {}& |n-2, n+1, \ldots, n+N-2, n+N| \\ & - |n-1, n, n+2, \ldots, n+N-2, n+N|,\end{aligned} \tag{2.15}$$

$$\begin{aligned}(\partial_s(\partial_y - 2c\partial_x + Nc^2) - 2)\tau_n = {}& |n-2, n+1, \ldots, n+N-2, n+N+1| \\ & - |n-1, n, n+2, \ldots, n+N-2, n+N+1| \\ & - |n-2, n+1, \ldots, n+N-3, n+N-1, n+N| \\ & + |n-1, n, n+2, \ldots, n+N-3, n+N-1, n+N|,\end{aligned} \tag{2.16}$$

which are proved by using the linear dispersion relations (2.3)–(2.6). (See the appendix.)

The first equation of equations (2.1)

Let us introduce an identity for $2N \times 2N$ determinant,

$$\begin{vmatrix} n-1 & n+1 & \cdots & n+N-2 & n+N & n & \emptyset & & & n+N-1 \\ n-1 & & \emptyset & & n+N & n & n+1 & \cdots & n+N-2 & n+N-1 \end{vmatrix} = 0.$$

Applying the Laplace expansion to the left-hand side, we obtain the algebraic bilinear identity for determinants,

$$\begin{aligned}&|n-1,n+1,\ldots,n+N-2,n+N|\times|n,n+1,\ldots,n+N-2,n+N-1|\\&\quad-|n,n+1,\ldots,n+N-2,n+N|\\&\quad\times|n-1,n+1,\ldots,n+N-2,n+N-1|\\&\quad+|n+1,\ldots,n+N-2,n+N-1,n+N|\\&\quad\times|n-1,n,n+1,\ldots,n+N-2|=0,\end{aligned} \tag{2.17}$$

which is rewritten by using (2.8), (2.9), (2.11) and (2.13) into the differential bilinear equation

$$(\partial_t(\partial_x-Nc)-1)\tau_n\times\tau_n-(\partial_x-Nc)\tau_n\times\partial_t\tau_n+\tau_{n+1}\tau_{n-1}=0,$$

i.e.,

$$(\partial_t\partial_x\tau_n-\tau_n)\,\tau_n-\partial_t\tau_n\partial_x\tau_n+\tau_{n+1}\tau_{n-1}=0.$$

Setting $n=0$, $f=\tau_0$, $g=\tau_1$, $h=\tau_{-1}$, the above bilinear equation leads to the first equation of (2.1).

The second equation of equations (2.1)

Let us introduce two identities for $2N\times 2N$ determinants,

$$\begin{vmatrix}n-1 & n+1 & \cdots & n+N-3 & n+N-1 & n+N & n & & \emptyset & \\ n-1 & & & \emptyset & & n+N & n & n+1 & \cdots & n+N-1\end{vmatrix}=0,$$

$$\begin{vmatrix}n-1 & n+1 & \cdots & n+N-2 & n+N+1 & & & \emptyset & & n+N-1\\ n-1 & & \emptyset & & n+N+1 & n & n+1 & \cdots & n+N-2 & n+N-1\end{vmatrix}=0.$$

Applying the Laplace expansion to the left-hand side, we obtain the algebraic bilinear identities for determinants,

$$\begin{aligned}&|n-1,n+1,\ldots,n+N-3,n+N-1,n+N|\times|n,n+1,\ldots,n+N-2,n+N-1|\\&\quad-|n,n+1,\ldots,n+N-3,n+N-1,n+N|\times|n-1,n+1,\ldots,n\\&\quad+N-2,n+N-1|+|n-1,n,n+1,\ldots,n+N-3,n+N-1|\\&\quad\times|n+1,n+2,\ldots,n+N-1,n+N|=0,\end{aligned} \tag{2.18}$$

$$\begin{aligned}&|n-1,n+1,\ldots,n+N-2,n+N+1|\times|n,n+1,\ldots,n+N-2,n+N-1|\\&\quad-|n,n+1,\ldots,n+N-2,n+N+1|\times|n-1,n+1,\ldots,n\\&\quad+N-2,n+N-1|+|n+1,\ldots,n+N-1,n+N+1|\\&\quad\times|n-1,n,n+1,\ldots,n+N-3,n+N-2|=0.\end{aligned} \tag{2.19}$$

Taking the difference of these two bilinear identities, it is rewritten by using (2.8)–(2.11) and (2.14) into the differential bilinear equation,

$$\begin{aligned}&\partial_t(\partial_y-2c\partial_x+Nc^2)\tau_n\times\tau_n-(\partial_y-2c\partial_x+Nc^2)\tau_n\times\partial_t\tau_n\\&\quad+(\partial_x-Nc)\tau_{n+1}\times\tau_{n-1}-\tau_{n+1}(\partial_x-Nc)\tau_{n-1}=0,\end{aligned}$$

i.e.,

$$(\partial_t\partial_y\tau_n)\tau_n-\partial_y\tau_n\partial_t\tau_n-2c((\partial_t\partial_x\tau_n)\tau_n-\partial_x\tau_n\partial_t\tau_n)+(\partial_x\tau_{n+1})\tau_{n-1}-\tau_{n+1}(\partial_x\tau_{n-1})=0,$$

which is nothing but the second equation of (2.1).

The third equation of equations (2.1)

Let us introduce two identities for $2N \times 2N$ determinants,

$$\begin{vmatrix} n-2 & n+1 & \cdots & n+N-2 & n+N & n & & \emptyset & & n+N-1 \\ & & \emptyset & & n+N & n & n+1 & \cdots & n+N-2 & n+N-1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} n-1 & n & n+2 & \cdots & n+N-2 & n+N & n+1 & & \emptyset & & n+N-1 \\ & & & \emptyset & & n+N & n+1 & n & n+2 & \cdots & n+N-2 & n+N-1 \end{vmatrix} = 0.$$

Applying the Laplace expansion to the left-hand side, we obtain the algebraic bilinear identities for determinants,

$$\begin{aligned} &|n-2, n+1, n+2, \ldots, n+N-2, n+N| \times |n, n+1, \ldots, n+N-2, n+N-1| \\ &\quad - |n-2, n+1, n+2, \ldots, n+N-2, n+N-1| \times |n, n+1, n+2, \ldots, n \\ &\quad + N-2, n+N| + |n-2, n, n+1, n+2, \ldots, n+N-2| \\ &\quad \times |n+1, n+2, \ldots, n+N-2, n+N-1, n+N| = 0, \end{aligned} \tag{2.20}$$

$$\begin{aligned} &|n-1, n, n+2, \ldots, n+N-2, n+N| \times |n, n+1, \ldots, n+N-2, n+N-1| \\ &\quad - |n-1, n, n+2, \ldots, n+N-2, n+N-1| \times |n, n+1, n+2, \ldots, n \\ &\quad + N-2, n+N| + |n-1, n, n+1, n+2, \ldots, n+N-2| \\ &\quad \times |n, n+2, \ldots, n+N-2, n+N-1, n+N| = 0. \end{aligned} \tag{2.21}$$

Taking the difference of these two bilinear identities, it is rewritten by using (2.12) and (2.15) into the differential bilinear equation,

$$\partial_s(\partial_x - Nc)\tau_n \times \tau_n - \partial_s\tau_n \times (\partial_x - Nc)\tau_n + (\partial_t\tau_{n-1})\tau_{n+1} - \tau_{n-1}(\partial_t\tau_{n+1}) = 0,$$

i.e.,

$$(\partial_s\partial_x\tau_n)\tau_n - \partial_s\tau_n\partial_x\tau_n - (\partial_t\tau_{n+1})\tau_{n-1} + \tau_{n+1}(\partial_t\tau_{n-1}) = 0.$$

Setting $n = 0$, $f = \tau_0$, $g = \tau_1$, $h = \tau_{-1}$, the above bilinear equation leads to the third equation of (2.1).

The fourth equation of equations (2.1)

Let us introduce four identities for $2N \times 2N$ determinants,

$$\begin{vmatrix} n-2 & n+1 & \cdots & n+N-2 & n+N+1 & n & & \emptyset & & \\ n-2 & & \emptyset & & n+N+1 & n & n+1 & \cdots & n+N-2 & n+N-1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} n-2 & n+1 & \cdots & n+N-3 & n+N-1 & n+N & n & & \emptyset & \\ n-2 & & & \emptyset & & n+N & n & n+1 & \cdots & n+N-1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} n-1 & n & n+2 & \cdots & n+N-2 & n+N+1 & n+1 & & \emptyset & & \\ n-1 & & & \emptyset & & n+N+1 & n+1 & n & n+2 & \cdots & n+N-1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} n-1 & n & n+2 & \cdots & n+N-3 & n+N-1 & n+N & n+1 & & \emptyset & & \\ n-1 & & & \emptyset & & & n+N & n+1 & n & n+2 & \cdots & n+N-1 \end{vmatrix} = 0.$$

Applying the Laplace expansion to the left-hand side, we obtain the algebraic bilinear identities for determinants,

$$\begin{aligned}|n-2, n+1, \ldots, n+N-2, n+N+1| \times |n, n+1, \ldots, n+N-2, n+N-1|\\ - |n, n+1, \ldots, n+N-2, n+N+1|\\ \times |n-2, n+1, \ldots, n+N-2, n+N-1|\\ + |n-2, n, n+1, \ldots, n+N-2|\\ \times |n+1, \ldots, n+N-2, n+N-1, n+N+1| = 0,\end{aligned} \tag{2.22}$$

$$\begin{aligned}|n-2, n+1, \ldots, n+N-3, n+N-1, n+N| \times |n, n+1, \ldots, n+N-1|\\ - |n, n+1, \ldots, n+N-3, n+N-1, n+N|\\ \times |n-2, n+1, \ldots, n+N-1|\\ + |n-2, n, n+1, \ldots, n+N-3, n+N-1|\\ \times |n+1, \ldots, n+N-1, n+N| = 0,\end{aligned} \tag{2.23}$$

$$\begin{aligned}|n-1, n, n+2, \ldots, n+N-2, n+N+1| \times |n, n+1, \ldots, n+N-1|\\ - |n, n+1, \ldots, n+N-2, n+N+1| \times |n-1, n, n+2, \ldots, n+N-1|\\ + |n-1, n, n+1, \ldots, n+N-2|\\ \times |n, n+2, \ldots, n+N-1, n+N+1| = 0,\end{aligned} \tag{2.24}$$

$$\begin{aligned}|n-1, n, n+2, \ldots, n+N-3, n+N-1, n+N| \times |n, n+1, \ldots, n+N-1|\\ - |n, n+1, \ldots, n+N-3, n+N-1, n+N|\\ \times |n-1, n, n+2, \ldots, n+N-1|\\ + |n-1, n, n+1, \ldots, n+N-3, n+N-1|\\ \times |n, n+2, \ldots, n+N-1, n+N| = 0.\end{aligned} \tag{2.25}$$

Taking an appropriate linear combination of these four bilinear identities, it is rewritten by using (2.16) into the differential bilinear equation,

$$\begin{aligned}(\partial_s(\partial_y - 2c\partial_x + Nc^2) - 2)\tau_n \times \tau_n - (\partial_y - 2c\partial_x + Nc^2)\tau_n \times \partial_s\tau_n + \partial_t\tau_{n-1} \times (\partial_x - Nc)\tau_{n+1}\\ - (\partial_t(\partial_x - Nc) - 1)\tau_{n-1} \times \tau_{n+1} - \tau_{n-1}(\partial_t(\partial_x - Nc) - 1)\tau_{n+1}\\ + (\partial_x - Nc)\tau_{n-1} \times \partial_t\tau_{n+1} = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}(\partial_s\partial_y\tau_n)\tau_n - \partial_y\tau_n\partial_s\tau_n - 2c((\partial_s\partial_x\tau_n)\tau_n - \partial_x\tau_n\partial_s\tau_n) - 2\tau_n\tau_n\\ - (\partial_t\partial_x\tau_{n+1})\tau_{n-1} + \partial_x\tau_{n+1}\partial_t\tau_{n-1} + \partial_t\tau_{n+1}\partial_x\tau_{n-1}\\ - \tau_{n+1}\partial_t\partial_x\tau_{n-1} + 2\tau_{n+1}\tau_{n-1} = 0,\end{aligned}$$

which leads to the fourth equation of (2.1). □

Theorem 2.2. *Bilinear equations*

$$\begin{cases} -\left(\frac{1}{2}D_tD_x - 1\right)f \cdot f = gh, \\ 2cff = (D_x + 2c)g \cdot h, \\ -2ff = (D_tD_x + 2cD_t - 2)g \cdot h, \end{cases} \tag{2.26}$$

have a determinant solution

$$f = \tau_0, \qquad g = \tau_1, \qquad h = \tau_{-1},$$

$$\tau_n = \begin{vmatrix} \psi_1^{(n)} & \psi_1^{(n+1)} & \cdots & \psi_1^{(n+N-1)} \\ \psi_2^{(n)} & \psi_2^{(n+1)} & \cdots & \psi_2^{(n+N-1)} \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)} & \psi_N^{(n+1)} & \cdots & \psi_N^{(n+N-1)} \end{vmatrix},$$

where

$$\psi_i^{(n)} = a_{i,1}(p_i - c)^n \, \mathrm{e}^{\xi_i} + a_{i,2}(-p_i - c)^n \, \mathrm{e}^{\eta_i},$$

$$\xi_i = p_i x + \frac{1}{p_i - c} t + \frac{1}{(p_i - c)^2} s + \xi_{i0},$$

$$\eta_i = -p_i x - \frac{1}{p_i + c} t + \frac{1}{(p_i + c)^2} s + \eta_{i0}.$$

Proof. In the previous lemma, apply the 2-reduction condition

$$q_i = -p_i.$$

This condition gives a constraint

$$D_y = 0$$

into bilinear equations. Thus we have

$$\begin{cases} \left(\frac{1}{2} D_t D_x - 1\right) f \cdot f = -gh, \\ c D_t D_x f \cdot f = D_x g \cdot h, \\ \frac{1}{2} D_s D_x f \cdot f = D_t g \cdot h, \\ (-c D_s D_x - 2) f \cdot f = (D_t D_x - 2) g \cdot h. \end{cases} \tag{2.27}$$

After simple manipulations, we have

$$\begin{cases} -\left(\frac{1}{2} D_t D_x - 1\right) f \cdot f = gh, \\ 2cff = (D_x + 2c) g \cdot h, \\ -2ff = (D_t D_x + 2c D_t - 2) g \cdot h. \end{cases} \tag{2.28}$$

Let us consider a determinant solution. When we apply the 2-reduction condition

$$q_i = -p_i$$

on the determinant solution in the previous lemma, we will have

$$\tau_n = \begin{vmatrix} \psi_1^{(n)} & \psi_1^{(n+1)} & \cdots & \psi_1^{(n+N-1)} \\ \psi_2^{(n)} & \psi_2^{(n+1)} & \cdots & \psi_2^{(n+N-1)} \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)} & \psi_N^{(n+1)} & \cdots & \psi_N^{(n+N-1)} \end{vmatrix},$$

where

$$\psi_i^{(n)} = a_{i,1}(p_i - c)^n \, \mathrm{e}^{\xi_i} + a_{i,2}(-p_i - c)^n \, \mathrm{e}^{\eta_i},$$

$$\xi_i = p_i x + \frac{1}{p_i - c} t + \frac{1}{(p_i - c)^2} s + \xi_{i0},$$

$$\eta_i = -p_i x - \frac{1}{p_i + c} t + \frac{1}{(p_i + c)^2} s + \eta_{i0}.$$

Thus the theorem is proved. □

Theorem 2.3. *The CH equation*

$$(\partial_T + w\partial_X)(w_{XX} - w) + 2w_X(w_{XX} - w - \kappa^2) = 0,$$

i.e.,

$$w_T + 2\kappa^2 w_X - w_{TXX} + 3ww_X = 2w_X w_{XX} + ww_{XXX},$$

where $\kappa^2 = 1/c$, is decomposed into bilinear equations

$$\begin{cases} -\left(\frac{1}{2}D_t D_x - 1\right) f \cdot f = gh, \\ 2cff = (D_x + 2c)g \cdot h, \\ -2ff = (D_t D_x + 2cD_t - 2)g \cdot h, \end{cases} \tag{2.29}$$

through the hodograph transformation

$$\begin{cases} X = 2cx + \log \dfrac{g}{h}, \\ T = t, \end{cases}$$

and the dependent variable transformation

$$w = \left(\log \frac{g}{h}\right)_t.$$

Proof. Consider the dependent variable transformation

$$u = \frac{g}{f}, \qquad v = \frac{h}{f}.$$

From bilinear equations (2.29), we obtain

$$\begin{cases} -((\log f)_{xt} - 1) = uv, \\ 2c = (D_x + 2c)u \cdot v, \\ -2 = (D_t D_x + (2\log f)_{xt} + 2cD_t - 2)u \cdot v. \end{cases} \tag{2.30}$$

Eliminating $(\log f)_{xt}$ from the third equation using the first equation of (2.30), we obtain

$$\begin{cases} 2c = (D_x + 2c)u \cdot v, \\ -2 = (D_t D_x + 2cD_t - 2uv)u \cdot v. \end{cases} \tag{2.31}$$

Thus we have

$$\begin{cases} 2c = \left(\left(\log \dfrac{u}{v}\right)_x + 2c\right) uv, \\ -2 = \left((\log uv)_{xt} + \left(\log \dfrac{u}{v}\right)_x \left(\log \dfrac{u}{v}\right)_t + 2c\left(\log \dfrac{u}{v}\right)_t - 2uv\right) uv. \end{cases} \tag{2.32}$$

Eliminating $\left(\log \frac{u}{v}\right)_x + 2c$ from the second equation using the first equation of (2.32), we have

$$\begin{cases} 2c = \left(\left(\log \dfrac{u}{v}\right)_x + 2c\right) uv, \\ -2 = \left((\log uv)_{xt} + \dfrac{2c}{uv}\left(\log \dfrac{u}{v}\right)_t - 2uv\right) uv. \end{cases}$$

Introducing new dependent variables

$$\phi = \frac{u}{v} = \frac{g}{h}, \qquad \rho = uv = \frac{gh}{f^2},$$

we have

$$\begin{cases} \dfrac{2c}{\rho} = (\log\phi)_x + 2c, \\ -2 = \rho(\log\rho)_{xt} + 2c(\log\phi)_t - 2\rho^2. \end{cases} \tag{2.33}$$

Let

$$w = (\log\phi)_t = \left(\log\frac{g}{h}\right)_t.$$

From the first equation of (2.33), we obtain

$$-(\log\rho)_t = \frac{(\log\phi)_{xt}}{2c + (\log\phi)_x} = \frac{\rho}{2c}(\log\phi)_{xt} = \frac{\rho}{2c}w_x.$$

Thus we have

$$\begin{cases} -(\log\rho)_t = \dfrac{\rho}{2c}w_x, \\ \dfrac{\rho}{2c}(\log\rho)_{xt} + w + \dfrac{1}{c} = \dfrac{1}{c}\rho^2. \end{cases} \tag{2.34}$$

Replacing $(\log\rho)_t$ in the second equation by $-\frac{\rho}{2c}w_x$ using the first equation, we obtain

$$\begin{cases} -(\log\rho)_t = \dfrac{\rho}{2c}w_x, \\ -\dfrac{\rho}{2c}\left(\dfrac{\rho}{2c}w_x\right)_x + w + \dfrac{1}{c} = \dfrac{1}{c}\rho^2. \end{cases}$$

Consider the hodograph transformation

$$\begin{cases} X = 2cx + \log\phi, \\ T = t. \end{cases}$$

Then we have

$$\frac{\partial X}{\partial x} = 2c + (\log\phi)_x = 2c + \left(\log\frac{g}{h}\right)_x = 2c\frac{f^2}{gh} = \frac{2c}{\rho},$$

$$\frac{\partial X}{\partial t} = (\log\phi)_t = \left(\log\frac{g}{h}\right)_t = w,$$

$$\begin{cases} \partial_x = \dfrac{2c}{\rho}\partial_X, \\ \partial_t = \partial_T + w\partial_X. \end{cases}$$

Using these results, we obtain

$$\begin{cases} -(\partial_T + w\partial_X)\log\rho = w_X, \\ -w_{XX} + w + \dfrac{1}{c} = \dfrac{1}{c}\rho^2. \end{cases} \tag{2.35}$$

Eliminating ρ from the first equation using the second equation of equations (2.35), we obtain

$$(\partial_T + w\partial_X)\log\left(-w_{XX} + w + \frac{1}{c}\right) = -2w_X.$$

Thus, we finally obtain the CH equation

$$(\partial_T + w\partial_X)(w_{XX} - w) + 2w_X\left(w_{XX} - w - \frac{1}{c}\right) = 0.$$

The theorem is proved. □

Furthermore, we have the following corollary.

Corollary 2.4. *The CH equation has a determinant form of N-soliton solutions.*

Proof. From theorems 2.2 and 2.3, the proof is obvious. □

3. A semi-discrete Camassa–Holm equation

Lemma 3.1. *Bilinear equations*

$$\begin{cases} \left(\frac{1-ac}{a}D_t - 1\right) f(k+1,l)\cdot f(k,l) = -g(k+1,l)h(k,l), \\ \left(\frac{1-bc}{b}D_t - 1\right) f(k,l+1)\cdot f(k,l) = -g(k,l+1)h(k,l), \\ \left(\frac{1-ac}{a}D_s - D_t\right) f(k+1,l)\cdot f(k,l) = D_t g(k+1,l)\cdot h(k,l), \\ \left(\frac{1-bc}{b}D_s - D_t\right) f(k,l+1)\cdot f(k,l) = D_t g(k,l+1)\cdot h(k,l), \end{cases} \tag{3.1}$$

have a determinant solution

$$f(k,l) = \tau_0(k,l), \qquad g(k,l) = \tau_1(k,l), \qquad h(k,l) = \tau_{-1}(k,l),$$

$$\tau_n(k,l) = \begin{vmatrix} \psi_1^{(n)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l) \end{vmatrix},$$

where

$$\psi_i^{(n)}(k,l) = a_{i,1}(p_i - c)^n(1-ap_i)^{-k}(1-bp_i)^{-l}e^{\xi_i} + a_{i,2}(q_i - c)^n(1-aq_i)^{-k}(1-bq_i)^{-l}e^{\eta_i},$$

$$\xi_i = \frac{1}{p_i - c}t + \frac{1}{(p_i - c)^2}s + \xi_{i0},$$

$$\eta_i = \frac{1}{q_i - c}t + \frac{1}{(q_i - c)^2}s + \eta_{i0}.$$

Proof. Consider the following Casorati determinant solution,

$$\tau_n(k,l) = \begin{vmatrix} \psi_1^{(n)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\ \psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l) \end{vmatrix},$$

where $\psi_i^{(n)}$'s are arbitrary functions of two continuous independent variables, t and s, and two discrete independent variables, k and l, which satisfy the linear dispersion relations,

$$\Delta_k \psi_i^{(n)} = \psi_i^{(n+1)} + c\psi_i^{(n)}, \tag{3.2}$$

$$\Delta_l \psi_i^{(n)} = \psi_i^{(n+1)} + c\psi_i^{(n)}, \tag{3.3}$$

$$\partial_t \psi_i^{(n)} = \psi_i^{(n-1)}, \tag{3.4}$$

$$\partial_s \psi_i^{(n)} = \psi_i^{(n-2)}, \tag{3.5}$$

where Δ_k and Δ_l are defined as $\Delta_k\psi(k,l) = \frac{\psi(k,l)-\psi(k-1,l)}{a}$ and $\Delta_l\psi(k,l) = \frac{\psi(k,l)-\psi(k,l-1)}{b}$, respectively. Thus we can choose $\psi_i^{(n)}$ as follows:

$$\psi_i^{(n)}(k,l) = (p_i - c)^n(1-ap_i)^{-k}(1-bp_i)^{-l}\,\mathrm{e}^{\xi_i} + (q_i - c)^n(1-aq_i)^{-k}(1-bq_i)^{-l}\,\mathrm{e}^{\eta_i},$$

$$\xi_i = \frac{1}{p_i - c}t + \frac{1}{(p_i - c)^2}s + \xi_{i0},$$

$$\eta_i = \frac{1}{q_i - c}t + \frac{1}{(q_i - c)^2}s + \eta_{i0}.$$

We use the following notation for simplicity:

$$\left|n_{1_{k_1}}, n_{2_{k_2}}, \ldots, n_{N_{k_N}}\right| = \begin{vmatrix} \psi_1^{(n_1)}(k_1 l_1) & \psi_1^{(n_2)}(k_2 l_2) & \cdots & \psi_1^{(n_N)}(k_N l_N) \\ \psi_2^{(n_1)}(k_1 l_1) & \psi_2^{(n_2)}(k_2 l_2) & \cdots & \psi_2^{(n_N)}(k_N l_N) \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n_1)}(k_1 l_1) & \psi_N^{(n_2)}(k_2 l_2) & \cdots & \psi_N^{(n_N)}(k_N l_N) \end{vmatrix}. \tag{3.6}$$

In this notation, $\tau_n(k,l)$ is rewritten as

$$\tau_n(k,l) = |n_{k,l}, n+1_{k,l}, \ldots, n+N-1_{k,l}|. \tag{3.7}$$

or suppressing the index k and l

$$\tau_n(k,l) = |n, n+1, \ldots, n+N-1|.$$

The first equation of equations (3.1)

By using the Casoratian technique developed in [37, 38], it is possible to derive the following differential and difference formulas for the τ-function:

$$\tau_{n-1}(k,l) = |n-1, n, \ldots, n+N-2|, \tag{3.8}$$

$$\begin{aligned}\tau_n(k+1,l) &= \frac{1}{(1-ac)^{N-2}}|n_{k+1}, n+1, \ldots, n+N-2, n+N-1_{k+1}| \\ &= \frac{1}{(1-ac)^{N-1}}|n, \ldots, n+N-2, n+N-1_{k+1}|,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\tau_{n+1}(k+1,l) &= \frac{1}{(1-ac)^{N-1}}|n+1, \ldots, n+N-1, n+N_{k+1}| \\ &= \frac{1}{a(1-ac)^{N-2}}|n+1, \ldots, n+N-1, n+N-1_{k+1}|,\end{aligned} \tag{3.10}$$

$$\partial_t\tau_n = |n-1, n+1, \ldots, n+N-2, n+N-1|, \tag{3.11}$$

$$\frac{1-ac}{a}\tau_n(k+1,l) = \frac{1}{a(1-ac)^{N-2}}|n, \ldots, n+N-2, n+N-1_{k+1}|, \tag{3.12}$$

$$\begin{aligned}
\frac{1-ac}{a}&\partial_t\tau_n(k+1,l) - \tau_n(k+1,l)\\
&= \frac{1-ac}{a(1-ac)^{N-2}}|n-1_{k+1}, n+1, \ldots, n+N-2, n+N-1_{k+1}|\\
&\quad - |n_{k+1}, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|\\
&= \frac{1}{a(1-ac)^{N-2}}|n-1, n+1, \ldots, n+N-2, n+N-1_{k+1}|\\
&\quad + \frac{1}{(1-ac)^{N-2}}|n_{k+1}, n+1, \ldots, n+N-2, n+N-1_{k+1}|\\
&\quad - \frac{1}{(1-ac)^{N-2}}|n_{k+1}, n+1, \ldots, n+N-2, n+N-1_{k+1}|\\
&= \frac{1}{a(1-ac)^{N-2}}|n-1, n+1, \ldots, n+N-2, n+N-1_{k+1}|.
\end{aligned} \tag{3.13}$$

Let us introduce an identity for $2N \times 2N$ determinant,

$$\begin{vmatrix} n+1_k & \cdots & n+N-2_k & n+N-1_k & n+N-1_{k+1} & n-1_k & n_k & & \emptyset & \\ & \emptyset & & n+N-1_k & n+N-1_{k+1} & n-1_k & n_k & n+1_k & \cdots & n+N-2_k \end{vmatrix} = 0.$$

Applying the Laplace expansion to the left-hand side, we obtain the algebraic bilinear identity for determinants,

$$\begin{aligned}
|n-1, n+1, \ldots, n+N-2, n+N-1_{k+1}| &\times |n, n+1, \ldots, n+N-2, n+N-1|\\
&- |n, n+1, \ldots, n+N-2, n+N-1_{k+1}|\\
&\times |n-1, n+1, \ldots, n+N-2, n+N-1|\\
&+ |n+1, \ldots, n+N-2, n+N-1, n+N-1_{k+1}|\\
&\times |n-1, n, n+1, \ldots, n+N-2| = 0,
\end{aligned} \tag{3.14}$$

which is rewritten into the differential bilinear equation

$$\begin{aligned}
\left(\frac{1-ac}{a}\partial_t\tau_n(k+1,l) - \tau_n(k+1,l)\right)\tau_n(k,l) &- \frac{1-ac}{a}\tau_n(k+1,l)\partial_t\tau_n(k,l)\\
&+ \tau_{n+1}(k+1,l)\tau_{n-1}(k,l) = 0.
\end{aligned}$$

Setting $n = 0$, $f = \tau_0$, $g = \tau_1$, $h = \tau_{-1}$, the above bilinear equation leads to the first equation of (3.1).

The second equation of equations (3.1)

The proof is similar to the proof of the first equation.

The third equation of equations (3.1)

We use the following differential and difference formulas for the τ-function:

$$\tau_{n-1}(k,l) = |n-1, n, \ldots, n+N-2|, \tag{3.15}$$

$$\tau_n(k+1,l) = |n_{k+1}, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|, \tag{3.16}$$

$$\tau_{n+1}(k+1,l) = |n+1_{k+1}, \ldots, n+N-1_{k+1}, n+N_{k+1}|$$
$$= -\frac{1}{a}|n+1_{k+1}, \ldots, n+N-1_{k+1}, n+N-1|, \tag{3.17}$$

$$\partial_t \tau_n(k,l) = |n-1, n+1, \ldots, n+N-2, n+N-1|, \tag{3.18}$$

$$\frac{1-ac}{a}\tau_n(k+1,l) = \frac{1}{a}|n, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|$$
$$= \frac{1}{a}|n_{k+1}, n+1, n+2_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|, \tag{3.19}$$

$$\partial_t \tau_n(k+1,l) = |n-1_{k+1}, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|, \tag{3.20}$$

$$\partial_t \tau_{n+1}(k+1,l) = |n_{k+1}, n+2_{k+1}, \ldots, n+N-1_{k+1}, n+N_{k+1}| \tag{3.21}$$

$$= -\frac{1}{a}|n_{k+1}, n+2_{k+1}, \ldots, n+N-1_{k+1}, n+N-1|, \tag{3.22}$$

$$\partial_t \tau_{n-1}(k,l) = |n-2, n, \ldots, n+N-3, n+N-2|, \tag{3.23}$$

$$\partial_s \tau_n(k,l) = |n-2, n+1, \ldots, n+N-2, n+N-1|$$
$$+ |n, n-1, n+2, \ldots, n+N-2, n+N-1|, \tag{3.24}$$

$$\frac{1-ac}{a}\partial_s \tau_n(k+1,l) = \frac{1-ac}{a}|n-2_{k+1}, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|$$
$$+ \frac{1-ac}{a}|n_{k+1}, n-1_{k+1}, n+2_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|$$
$$= |n-1_{k+1}, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|$$
$$+ \frac{1}{a}|n-2, n+1_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|$$
$$- \frac{1}{a}|n-1, n_{k+1}, n+2_{k+1}, \ldots, n+N-2_{k+1}, n+N-1_{k+1}|. \tag{3.25}$$

Let us introduce following identities for $2N \times 2N$ determinant:

$$\begin{vmatrix} n-1_k & n_{k+1} & n+2_{k+1} & \cdots & n+N-1_{k+1} & n_k & n+1_k & & \emptyset & & n+N-1_k \\ n-1_k & & & \emptyset & & n_k & n+1_k & n+2_k & \cdots & n+N-2_k & n+N-1_k \end{vmatrix} = 0$$

and

$$\begin{vmatrix} n-2_k & n+1_{k+1} & \cdots & n+N-1_{k+1} & n_k & & \emptyset & & n+N-1_k \\ n-2_k & & \emptyset & & n_k & n+1_k & \cdots & n+N-2_k & n+N-1_k \end{vmatrix} = 0.$$

Applying the Laplace expansion to the left-hand side, we obtain the algebraic bilinear identities for determinants,

$$|n-1, n_{k+1}, n+2_{k+1}, \ldots, n+N-1_{k+1}| \times |n, n+1, n+2, \ldots, n+N-2, n+N-1|$$
$$- |n, n_{k+1}, n+2_{k+1}, \ldots, n+N-1_{k+1}|$$
$$\times |n-1, n+1, \ldots, n+N-2, n+N-1|$$
$$+ |n+1, n_{k+1}, n+2_{k+1}, \ldots, n+N-1_{k+1}|$$
$$\times |n-1, n, n+2, \ldots, n+N-2, n+N-1|$$
$$- |n_{k+1}, n+2_{k+1}, \ldots, n+N-1_{k+1}, n+N-1|$$
$$\times |n-1, n, n+1, n+2, \ldots, n+N-2| = 0 \tag{3.26}$$

and

$$\begin{aligned}
&|n-2, n+1_{k+1}, \ldots, n+N-1_{k+1}| \times |n, n+1, n+2, \ldots, n+N-2, n+N-1| \\
&\quad - |n, n+1_{k+1}, \ldots, n+N-1_{k+1}| \times |n-2, n+1, \ldots, n+N-2, n+N-1| \\
&\quad + |n+1_{k+1}, \ldots, n+N-1_{k+1}, n+N-1| \times |n-2, n, n+1, \ldots, n+N-2| \\
&= 0. \qquad (3.27)
\end{aligned}$$

By dividing the difference of the above two equations by a, we arrive at a bilinear equation

$$\begin{aligned}
&\frac{1-ac}{a}\partial_s\tau_n(k+1,l)\tau_n(k,l) - \frac{1-ac}{a}\tau_n(k+1,l)\partial_s\tau_n(k,l) \\
&\quad - \partial_t\tau_n(k+1,l)\tau_n(k,l) + \tau_n(k+1,l)\partial_t\tau_n(k,l) \\
&\quad - \partial_t\tau_{n+1}(k+1,l)\tau_{n-1}(k,l) + \tau_{n+1}(k+1,l)\partial_t\tau_{n-1}(k,l) = 0.
\end{aligned}$$

Setting $n=0$, $f=\tau_0$, $g=\tau_1$, $h=\tau_{-1}$, the above bilinear equation leads to the third equation of (3.1).

The fourth equation of equations (3.1)

The proof is similar to the proof of the third equation. □

Theorem 3.2. *Bilinear equations*

$$\begin{cases}
\left(\frac{1-ac}{a}D_t - 1\right) f_{k+1}\cdot f_k = -g_{k+1}h_k, \\
\left(\frac{1+ac}{a}D_t - 1\right) f_{k+1}\cdot f_k = -g_k h_{k+1}, \\
\left(\frac{1-ac}{a}D_s - D_t\right) f_{k+1}\cdot f_k = D_t g_{k+1}\cdot h_k, \\
\left(\frac{1+ac}{a}D_s + D_t\right) f_{k+1}\cdot f_k = D_t g_k\cdot h_{k+1},
\end{cases} \qquad (3.28)$$

have a determinant solution

$$f(k,l) = \tau_0(k,0), \qquad g(k,l) = \tau_1(k,0), \qquad h(k,l) = \tau_{-1}(k,0),$$

$$\tau_n(k,l) = \begin{vmatrix}
\psi_1^{(n)}(k,l) & \psi_1^{(n+1)}(k,l) & \cdots & \psi_1^{(n+N-1)}(k,l) \\
\psi_2^{(n)}(k,l) & \psi_2^{(n+1)}(k,l) & \cdots & \psi_2^{(n+N-1)}(k,l) \\
\vdots & \vdots & & \vdots \\
\psi_N^{(n)}(k,l) & \psi_N^{(n+1)}(k,l) & \cdots & \psi_N^{(n+N-1)}(k,l)
\end{vmatrix},$$

where

$$\begin{aligned}
\psi_i^{(n)}(k,l) &= a_{i,1}(p_i-c)^n(1-ap_i)^{-k}(1+ap_i)^{-l}\,\mathrm{e}^{\xi_i} \\
&\quad + a_{i,2}(-p_i-c)^n(1+ap_i)^{-k}(1-ap_i)^{-l}\,\mathrm{e}^{\eta_i},
\end{aligned}$$

$$\xi_i = \frac{1}{p_i-c}t + \frac{1}{(p_i-c)^2}s + \xi_{i0},$$

$$\eta_i = -\frac{1}{p_i+c}t + \frac{1}{(p_i+c)^2}s + \eta_{i0}.$$

Proof. Applying the 2-reduction condition

$$q_i = -p_i, \qquad b = -a,$$

the τ-function satisfies

$$\tau_n(k+1, l+1) = \frac{1}{\prod_{i=1}^{N}(1 - ap_i)(1 + ap_i)} \tau_n(k, l).$$

Let

$$f_k = f(k, 0), \qquad g_k = g(k, 0), \qquad h_k = h(k, 0),$$

then

$$\begin{cases} \left(\frac{1 - ac}{a} D_t - 1\right) f_{k+1} \cdot f_k = -g_{k+1} h_k, \\ \left(\frac{1 + ac}{a} D_t - 1\right) f_{k+1} \cdot f_k = -g_k h_{k+1}, \\ \left(\frac{1 - ac}{a} D_s - D_t\right) f_{k+1} \cdot f_k = D_t g_{k+1} \cdot h_k, \\ \left(\frac{1 + ac}{a} D_s + D_t\right) f_{k+1} \cdot f_k = D_t g_k \cdot h_{k+1}. \end{cases}$$

Let us consider a determinant solution. When we apply the 2-reduction condition

$$q_i = -p_i, \qquad b = -a$$

on the determinant solution in the previous lemma, we will have

$$\tau_n(k, l) = \begin{vmatrix} \psi_1^{(n)}(k, l) & \psi_1^{(n+1)}(k, l) & \cdots & \psi_1^{(n+N-1)}(k, l) \\ \psi_2^{(n)}(k, l) & \psi_2^{(n+1)}(k, l) & \cdots & \psi_2^{(n+N-1)}(k, l) \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n)}(k, l) & \psi_N^{(n+1)}(k, l) & \cdots & \psi_N^{(n+N-1)}(k, l) \end{vmatrix},$$

where

$$\psi_i^{(n)}(k, l) = a_{i,1}(p_i - c)^n (1 - ap_i)^{-k} (1 + ap_i)^{-l} \mathrm{e}^{\xi_i} + a_{i,2}(-p_i - c)^n (1 + ap_i)^{-k} (1 - ap_i)^{-l} \mathrm{e}^{\eta_i},$$

$$\xi_i = \frac{1}{p_i - c} t + \frac{1}{(p_i - c)^2} s + \xi_{i0},$$

$$\eta_i = -\frac{1}{p_i + c} t + \frac{1}{(p_i + c)^2} s + \eta_{i0}.$$

Thus the theorem is proved. □

We propose a semi-discrete analogue of the CH equation

$$\begin{cases} \Delta^2 w_k = \frac{1}{\delta_k} M \left(\delta_k M w_k + \frac{\kappa^2}{\delta_k} \frac{\kappa^4 \delta_k^2 - 4a^2}{\kappa^4 - a^2}\right), \\ \partial_t \delta_k = \left(1 - \frac{\delta_k^2}{4}\right) \delta_k \Delta w_k, \end{cases}$$

where a difference operator Δ and an average operator M are defined as

$$\Delta F_k = \frac{F_{k+1} - F_k}{\delta_k}, \qquad M F_k = \frac{F_{k+1} + F_k}{2}.$$

Theorem 3.3. *The semi-discrete CH equation*

$$\begin{cases} \Delta^2 w_k = \dfrac{1}{\delta_k} M\left(\delta_k M w_k + \dfrac{\kappa^2}{\delta_k}\dfrac{\kappa^4\delta_k^2 - 4a^2}{\kappa^4 - a^2}\right), \\ \partial_t \delta_k = \left(1 - \dfrac{\delta_k^2}{4}\right)\delta_k \Delta w_k \end{cases} \tag{3.29}$$

is decomposed into bilinear equations

$$\begin{cases} \left(\dfrac{1-ac}{a}D_t - 1\right) f_{k+1}\cdot f_k = -g_{k+1}h_k, \\ \left(\dfrac{1+ac}{a}D_t - 1\right) f_{k+1}\cdot f_k = -g_k h_{k+1}, \\ \left(\dfrac{1-ac}{a}D_s - D_t\right) f_{k+1}\cdot f_k = D_t g_{k+1}\cdot h_k, \\ \left(\dfrac{1+ac}{a}D_s + D_t\right) f_{k+1}\cdot f_k = D_t g_k\cdot h_{k+1}, \end{cases} \tag{3.30}$$

through the transformation

$$\delta_k = \frac{4a}{(\kappa^2 + a)\frac{g_{k+1}h_k}{f_{k+1}f_k} + (\kappa^2 - a)\frac{g_k h_{k+1}}{f_{k+1}f_k}}$$

and

$$w_k = \left(\log\frac{g_k}{h_k}\right)_t,$$

where $\kappa^2 = 1/c$.

Proof. Let us start from bilinear equations (3.30). By simple manipulation of equations (3.30), we obtain

$$\begin{cases} -2\left(\dfrac{1}{a}D_t - 1\right) f_{k+1}\cdot f_k = g_{k+1}h_k + g_k h_{k+1}, \\ 2acf_{k+1}f_k = (1+ac)g_{k+1}h_k - (1-ac)g_k h_{k+1}, \\ -2af_{k+1}f_k = ((1+ac)D_t - a)g_{k+1}\cdot h_k - ((1-ac)D_t + a)g_k\cdot h_{k+1}. \end{cases}$$

Let

$$u_k = \frac{g_k}{f_k}, \qquad v_k = \frac{h_k}{f_k}.$$

Then we have

$$\begin{cases} -2\left(\dfrac{1}{a}\left(\log\dfrac{f_{k+1}}{f_k}\right)_t - 1\right) = u_{k+1}v_k + u_k v_{k+1}, \\ 2ac = (1+ac)u_{k+1}v_k - (1-ac)u_k v_{k+1}, \\ -2a = \left[(1+ac)\left(\left(\log\dfrac{u_{k+1}}{v_k}\right)_t + \left(\log\dfrac{f_{k+1}}{f_k}\right)_t\right) - a\right]u_{k+1}v_k \\ \quad - \left[(1-ac)\left(\left(\log\dfrac{u_k}{v_{k+1}}\right)_t - \left(\log\dfrac{f_{k+1}}{f_k}\right)_t\right) + a\right]u_k v_{k+1}. \end{cases} \tag{3.31}$$

Substituting the first equation into the third equation of equations (3.31), we obtain

$$\begin{cases} 2ac = (1+ac)u_{k+1}v_k - (1-ac)u_k v_{k+1}, \\ -2a = (1+ac)u_{k+1}v_k \left(\log \dfrac{u_{k+1}}{v_k}\right)_t - (1-ac)u_k v_{k+1} \left(\log \dfrac{u_k}{v_{k+1}}\right)_t \\ \quad -\dfrac{a}{2}(u_{k+1}v_k + u_k v_{k+1})((1+ac)u_{k+1}v_k + (1-ac)u_k v_{k+1}) \\ \quad +a^2 c(u_{k+1}v_k - u_k v_{k+1}). \end{cases}$$

Simplifying the above equations, we have

$$\begin{cases} 2ac = (1+ac)u_{k+1}v_k - (1-ac)u_k v_{k+1}, \\ -2a = \dfrac{1}{2}((1+ac)u_{k+1}v_k + (1-ac)u_k v_{k+1}) \left(\log \dfrac{u_{k+1}v_{k+1}}{u_k v_k}\right)_t \\ \quad +ac \left(\log \dfrac{u_{k+1}u_k}{v_{k+1}v_k}\right)_t - 2au_{k+1}v_{k+1}u_k v_k. \end{cases} \tag{3.32}$$

Let

$$\phi_k = \frac{u_k}{v_k} = \frac{g_k}{h_k}, \qquad \rho_k = u_k v_k = \frac{g_k h_k}{f_k^2}.$$

From the first equation of (3.32), we obtain

$$\frac{2ac}{u_k v_{k+1}} = (1+ac)\frac{\phi_{k+1}}{\phi_k} - (1-ac), \qquad \frac{2ac}{u_{k+1}v_k} = (1+ac) - (1-ac)\frac{\phi_k}{\phi_{k+1}}.$$

Multiplying these two equations, we obtain

$$\frac{(2ac)^2}{\rho_{k+1}\rho_k} = \left((1+ac)\frac{\phi_{k+1}}{\phi_k} - (1-ac)\right)\left((1+ac) - (1-ac)\frac{\phi_k}{\phi_{k+1}}\right).$$

From the second equation of (3.32), we obtain

$$-2a = \frac{1}{2}((1+ac)u_{k+1}v_k + (1-ac)u_k v_{k+1})\left(\log \frac{\rho_{k+1}}{\rho_k}\right)_t + ac\,(\log \phi_{k+1}\phi_k)_t - 2a\rho_{k+1}\rho_k.$$

Thus we have

$$\frac{(1+ac)u_{k+1}v_k + (1-ac)u_k v_{k+1}}{4ac}\left(\log \frac{\rho_{k+1}}{\rho_k}\right)_t + \frac{1}{2}(\log \phi_{k+1}\phi_k)_t + \frac{1}{c} = \frac{1}{c}\rho_{k+1}\rho_k.$$

Let us define a lattice parameter

$$\delta_k = \frac{4ac}{(1+ac)u_{k+1}v_k + (1-ac)u_k v_{k+1}}.$$

Then

$$\delta_k = 2\frac{(1+ac)u_{k+1}v_k - (1-ac)u_k v_{k+1}}{(1+ac)u_{k+1}v_k + (1-ac)u_k v_{k+1}} = 2\frac{(1+ac)g_{k+1}h_k - (1-ac)g_k h_{k+1}}{(1+ac)g_{k+1}h_k + (1-ac)g_k h_{k+1}}$$

$$= 2\frac{(1+ac)\phi_{k+1} - (1-ac)\phi_k}{(1+ac)\phi_{k+1} + (1-ac)\phi_k} = 2\frac{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} + 1}.$$

Thus we have

$$\frac{\phi_{k+1}}{\phi_k} = \frac{1-ac}{1+ac}\frac{1+\frac{\delta_k}{2}}{1-\frac{\delta_k}{2}},$$

where a lattice parameter δ_k is a function depending on (k, t). The lattice parameter corresponds to $\frac{\partial X}{\partial x} = \frac{2c}{\rho} = 2c + (\log\phi)_x$ in the continuous case. At time t, $X = X_0 + \sum_{k=0}^{K-1}\delta_k$ is the x-coordinate of the k-th lattice point. Thus we have the following system:

$$\begin{cases} \dfrac{(2ac)^2}{\rho_{k+1}\rho_k} = \left((1+ac)\dfrac{\phi_{k+1}}{\phi_k} - (1-ac)\right)\left((1+ac) - (1-ac)\dfrac{\phi_k}{\phi_{k+1}}\right), \\ \dfrac{1}{\delta_k}\left(\log\dfrac{\rho_{k+1}}{\rho_k}\right)_t + \dfrac{1}{2}(\log\phi_{k+1}\phi_k)_t + \dfrac{1}{c} = \dfrac{1}{c}\rho_{k+1}\rho_k, \\ \delta_k = 2\dfrac{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} + 1}. \end{cases} \tag{3.33}$$

Let

$$w_k = (\log\phi_k)_t = \left(\log\frac{g_k}{h_k}\right)_t.$$

From the first equation of (3.33), we obtain

$$-(\log\rho_{k+1}\rho_k)_t = \frac{\frac{1+ac}{1-ac}\left(\frac{\phi_{k+1}}{\phi_k}\right)_t}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1} + \frac{-\left(\frac{\phi_k}{\phi_{k+1}}\right)_t}{\frac{1+ac}{1-ac} - \frac{\phi_k}{\phi_{k+1}}}$$
$$= \left(\frac{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k}}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1} + \frac{\frac{\phi_k}{\phi_{k+1}}}{\frac{1+ac}{1-ac} - \frac{\phi_k}{\phi_{k+1}}}\right)\left(\log\frac{\phi_{k+1}}{\phi_k}\right)_t$$
$$= \frac{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} + 1}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1}\left(\log\frac{\phi_{k+1}}{\phi_k}\right)_t$$
$$= \frac{2}{\delta_k}(w_{k+1} - w_k).$$

From the third equation of (3.33), we obtain

$$\frac{2}{\delta_k} + 1 = \frac{2\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k}}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1}, \qquad \frac{2}{\delta_k} - 1 = \frac{2}{\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1}.$$

Multiplying these two equations, we obtain

$$\frac{4}{\delta_k^2} - 1 = \frac{4}{\left(\frac{1+ac}{1-ac}\frac{\phi_{k+1}}{\phi_k} - 1\right)\left(1 - \frac{1-ac}{1+ac}\frac{\phi_k}{\phi_{k+1}}\right)} = \frac{4(1-ac)(1+ac)}{\frac{(2ac)^2}{\rho_{k+1}\rho_k}} = \left(\frac{1}{a^2c^2} - 1\right)\rho_{k+1}\rho_k.$$

Thus we have a system

$$\begin{cases} -(\log \rho_{k+1}\rho_k)_t = 2\dfrac{w_{k+1} - w_k}{\delta_k}, \\ \dfrac{1}{\delta_k}\left(\log\dfrac{\rho_{k+1}}{\rho_k}\right)_t + \dfrac{w_{k+1} + w_k}{2} + \dfrac{1}{c} = \dfrac{1}{c}\rho_{k+1}\rho_k, \\ \dfrac{4}{\delta_k^2} - 1 = \left(\dfrac{1}{a^2c^2} - 1\right)\rho_{k+1}\rho_k. \end{cases}$$

Let

$$r_k = \log \rho_k = \log \frac{g_k h_k}{f_k^2}.$$

Then we have

$$\begin{cases} -\partial_t(r_{k+1} + r_k) = 2\dfrac{w_{k+1} - w_k}{\delta_k}, \\ \dfrac{1}{\delta_k}\partial_t(r_{k+1} - r_k) + \dfrac{w_{k+1} + w_k}{2} + \dfrac{1}{c}\dfrac{1 - \frac{4a^2c^2}{\delta_k^2}}{1 - a^2c^2} = 0, \\ \dfrac{4}{\delta_k^2} - 1 = \left(\dfrac{1}{a^2c^2} - 1\right)\mathrm{e}^{r_{k+1}+r_k}. \end{cases}$$

Let

$$r_k' = \partial_t r_k.$$

Then

$$\begin{cases} -(r_{k+1}' + r_k') = 2\dfrac{w_{k+1} - w_k}{\delta_k}, \\ \dfrac{1}{\delta_k}(r_{k+1}' - r_k') + \dfrac{w_{k+1} + w_k}{2} + \dfrac{1}{c}\dfrac{1 - \frac{4a^2c^2}{\delta_k^2}}{1 - a^2c^2} = 0, \\ \partial_t\delta_k = \left(1 - \dfrac{\delta_k^2}{4}\right)(w_{k+1} - w_k). \end{cases}$$

Eliminating r_k', we have

$$\begin{cases} -2\left(\dfrac{w_{k+1} - w_k}{\delta_k} - \dfrac{w_k - w_{k-1}}{\delta_{k-1}}\right) + \delta_k\dfrac{w_{k+1} + w_k}{2} + \dfrac{\delta_k}{c}\dfrac{1 - \frac{4a^2c^2}{\delta_k^2}}{1 - a^2c^2} \\ \quad + \delta_{k-1}\frac{w_k + w_{k-1}}{2} + \frac{\delta_{k-1}}{c}\frac{1 - \frac{4a^2c^2}{\delta_{k-1}^2}}{1 - a^2c^2} = 0, \\ \partial_t\delta_k = \left(1 - \dfrac{\delta_k^2}{4}\right)(w_{k+1} - w_k). \end{cases} \tag{3.34}$$

This is a semi-discrete Camassa–Holm equation. Note that the lattice parameter depends on the time and space. □

Differentiating the first equation of (3.34) with respect to t, we obtain

$$-2\left(\frac{w_{k+1}-w_k}{\delta_k}-\frac{w_k-w_{k-1}}{\delta_{k-1}}\right)_t+\delta_k\frac{\partial_t w_{k+1}+\partial_t w_k}{2}+\delta_{k-1}\frac{\partial_t w_k+\partial_t w_{k-1}}{2}$$
$$+\left(1-\frac{\delta_k^2}{4}\right)(w_{k+1}-w_k)\left(\frac{w_{k+1}+w_k}{2}+\frac{1}{c}\frac{1+\frac{4a^2c^2}{\delta_k^2}}{1-a^2c^2}\right)$$
$$+\left(1-\frac{\delta_{k-1}^2}{4}\right)(w_k-w_{k-1})\left(\frac{w_k+w_{k-1}}{2}+\frac{1}{c}\frac{1+\frac{4a^2c^2}{\delta_{k-1}^2}}{1-a^2c^2}\right)=0.$$

Taking a continuous limit, this leads to the CH equation.

Then we have the following corollary.

Corollary 3.4. *The semi-discrete CH equation has a determinant form of N-soliton solutions.*

Proof. From theorems 3.2 and 3.3, the proof is obvious. □

4. Peakons, solitons and cuspons in the semi-discrete CH equation

4.1. Peakon limit in the semi-discrete Camassa–Holm equation

It is known that the CH equation has a peakon solution in the limit $\kappa \to 0$. Here we consider a peakon limit in the semi-discrete CH equation (3.29).

In the semi-discrete CH equation (3.29), consider the limit

$$\kappa \to 0, \qquad \frac{\kappa^2}{\delta_k} \to \delta(k),$$

where $\delta(k)$ is a dirac delta function. In this limit, the first equation of the semi-discrete CH equation (3.29) leads to

$$\partial_X^2 w - w = \delta(X - cT).$$

This differential equation has a solution

$$w = \sum_{i=1}^{N} A_i(T)\exp|X - cT|,$$

which is a form of the peakon solution.

Thus if we consider very small κ with very small δ_k at some points k, the solutions of the semi-discrete CH equation tend to the peakon solutions of the CH equation.

4.2. 1-soliton/cuspon solution

From the determinant formula with $a_{i,1}/a_{i,2} = \pm 1$, the τ-functions for 1-soliton/cuspon solution are

$$g \propto 1 \pm \left(\frac{c-p}{c+p}\right)\mathrm{e}^{\theta}, \qquad h \propto 1 \pm \left(\frac{c+p}{c-p}\right)\mathrm{e}^{\theta}, \tag{4.1}$$

with $\theta = 2p(x - vt - x_0)$, $v = 1/(c^2 - p^2)$ where $c = 1/\kappa^2 > 0$. This leads to a solution

$$w(x,t) = \frac{4p^2cv}{(c^2+p^2) \pm (c^2-p^2)\cosh\theta}, \tag{4.2}$$

$$X = 2cx + \log\left(\frac{g}{h}\right), \qquad T = t, \tag{4.3}$$

where the positive case of equation (4.2) stands for one smooth soliton solution when $p < c$, while the negative case of equation (4.2) stands for a 1-cuspon solution when $p > c$. Otherwise, the solution is singular. Thus, equation (4.2) for nonsingular cases can be expressed as

$$w(x,t) = \frac{4p^2cv}{(c^2+p^2) + |c^2-p^2|\cosh\theta}. \tag{4.4}$$

Similarly, for the semi-discrete case, we have

$$g_k \propto 1 + \left|\frac{c-p}{c+p}\right|\left(\frac{1+ap}{1-ap}\right)^k e^{\theta}, \qquad h_k \propto 1 + \left|\frac{c+p}{c-p}\right|\left(\frac{1+ap}{1-ap}\right)^k e^{\theta}, \tag{4.5}$$

with $\theta = -2pv(t+x_0)$, resulting in a solution of the form

$$w_k(t) = \frac{4p^2cv}{(c^2+p^2) + |c^2-p^2|\left[\left(\frac{1+ap}{1-ap}\right)^{-k} e^{-\theta} + \left(\frac{1+ap}{1-ap}\right)^{k} e^{\theta}\right]}, \tag{4.6}$$

in conjunction with a transform between an equidistance mesh (a) and a non-equidistance mesh

$$\delta_k = 2\frac{(1+ac)g_{k+1}h_k - (1-ac)g_kh_{k+1}}{(1+ac)g_{k+1}h_k + (1-ac)g_kh_{k+1}}. \tag{4.7}$$

Equation (4.6) corresponds to the 1-soliton solution when $p < c$ and the 1-cuspon solution when $p > c$.

4.3. 2-soliton/cuspon solutions

From the determinant formula with $a_{i,1}/a_{i,2} = \pm 1$, the τ-functions for the 2-soliton/cuspon solutions are

$$g \propto 1 + \left|\frac{c_1-p_1}{c_1+p_1}\right| e^{\theta_1} + \left|\frac{c_2-p_2}{c_2+p_2}\right| e^{\theta_2} + \left|\frac{(c_1-p_1)(c_2-p_2)}{(c_1+p_1)(c_2+p_2)}\right| \left(\frac{p_1-p_2}{p_1+p_2}\right)^2 e^{\theta_1+\theta_2},$$

$$h \propto 1 + \left|\frac{c_1+p_1}{c_1-p_1}\right| e^{\theta_1} + \left|\frac{c_2+p_2}{c_2-p_2}\right| e^{\theta_2} + \left|\frac{(c_1+p_1)(c_2+p_2)}{(c_1-p_1)(c_2-p_2)}\right| \left(\frac{p_1-p_2}{p_1+p_2}\right)^2 e^{\theta_1+\theta_2},$$

with $\theta_1 = 2p_1(x - v_1t - x_{10})$, $\theta_2 = 2p_2(x - v_2t - x_{20})$, $v_1 = 1/\left(c_1^2 - p_1^2\right)$, $v_2 = 1/\left(c_2^2 - p_2^2\right)$. The parametric solution can be calculated through

$$w(x,t) = \left(\log\frac{g}{h}\right)_t, \qquad X = 2cx + \log\left(\frac{g}{h}\right), \qquad T = t, \tag{4.8}$$

whose form is complicated and is omitted here. Note that the above expression includes the 2-soliton solution ($p_1 < c_1$, $p_2 < c_2$), the 2-cuspon solution ($p_1 > c_1$, $p_2 > c_2$) or the soliton–cuspon solution ($p_1 < c_1$, $p_2 > c_2$).

Similarly, for the semi-discrete case, we have

$$g_k \propto 1 + \left|\frac{c_1-p_1}{c_1+p_1}\right|\left(\frac{1+ap_1}{1-ap_1}\right)^k e^{\theta_1} + \left|\frac{c_2-p_2}{c_2+p_2}\right|\left(\frac{1+ap_2}{1-ap_2}\right)^k e^{\theta_2}$$
$$+ \left|\frac{(c_1-p_1)(c_2-p_2)}{(c_1+p_1)(c_2+p_2)}\right|\left(\frac{p_1-p_2}{p_1+p_2}\right)^2\left(\frac{1+ap_1}{1-ap_1}\right)^k\left(\frac{1+ap_2}{1-ap_2}\right)^k e^{\theta_1+\theta_2},$$

$$h_k \propto 1 + \left|\frac{c_1+p_1}{c_1-p_1}\right| \left(\frac{1+ap_1}{1-ap_1}\right)^k \mathrm{e}^{\theta_1} + \left|\frac{c_2+p_2}{c_2-p_2}\right| \left(\frac{1+ap_2}{1-ap_2}\right)^k \mathrm{e}^{\theta_2}$$
$$+ \left|\frac{(c_1+p_1)(c_2+p_2)}{(c_1-p_1)(c_2-p_2)}\right| \left(\frac{p_1-p_2}{p_1+p_2}\right)^2 \left(\frac{1+ap_1}{1-ap_1}\right)^k \left(\frac{1+ap_2}{1-ap_2}\right)^k \mathrm{e}^{\theta_1+\theta_2},$$

with $\theta_1 = 2p_1(-v_1t - x_{10})$, $\theta_2 = 2p_2(-v_2t - x_{20})$. The solution can be calculated through

$$w_k(t) = \left(\log \frac{g_k}{h_k}\right)_t, \tag{4.9}$$

with a transform

$$\delta_k = 2\frac{(1+ac)g_{k+1}h_k - (1-ac)g_kh_{k+1}}{(1+ac)g_{k+1}h_k + (1-ac)g_kh_{k+1}}. \tag{4.10}$$

The form is complicated and is omitted here.

5. Numerical computations

In this section, several examples will be illustrated to show that the integrable semi-discretization of the CH equation is a powerful scheme for the numerical solutions of the CH equation. They include (1) propagation of the 1-cuspon solution, (2) interaction of the 2-cuspon solutions and (3) head-on collision of the soliton–cuspon. In actual computations, the initial mesh spacing δ_k is assigned by

$$\delta_k = 2\frac{(1+ac)\phi_{k+1} - (1-ac)\phi_k}{(1+ac)\phi_{k+1} + (1-ac)\phi_k}, \tag{5.1}$$

where $\phi_k = g_k/h_k$ is obtainable from the corresponding determinant solutions. At each time step, from the second equation of (3.29), the evolution of δ_k can be exactly calculated by

$$\delta_k^{n+1} = 2\frac{c_k^n\,\mathrm{e}^{(w_{k+1}^n - w_k^n)} - 1}{c_k^n\,\mathrm{e}^{(w_{k+1}^n - w_k^n)} + 1}, \tag{5.2}$$

with $c_k^n = \left(2+\delta_k^n\right)/\left(2-\delta_k^n\right)$. Then, the solution w_k^{n+1} is easily computed by solving a tridiagonal linear system based on the first equation of the scheme. Therefore, the computation cost is less than other traditional numerical methods. Furthermore, we comment that the mesh spacing δ_k is driven automatically by the solution during the numerical computation. Therefore, we would like to call it the self-adaptive method. As shown subsequently, numerical solutions with high accuracy are obtained.

Example 1. 1-cuspon propagation. The parameters taken for the 1-cuspon solution are $p = 10.98$, $c = 10.0$. The number of grid is taken as 100 in an interval of width 4 in the *x*-domain, which implies a mesh size of $h = 0.04$. However, through the hodograph transformation, this corresponds to an interval of width 74.34 in the *X*-domain, which implies an average mesh size of 0.7434. In subsequent examples except for example 4, the grid number is fixed to be 100. The time-step size is taken as $\Delta t = 0.0004$. Figure 1(*a*) shows the initial condition. Figures 1(*b*) and (*c*) display the numerical solutions (solid line) and exact solutions (dotted line) at $t = 2, 4$, respectively. The L_∞ norm are 0.0365 at $t = 2$, and 0.0985 at $t = 4$. It is noted that the numerical error is mainly due to the numerical dispersion. In other words, even after a fairly long time, the numerical solution of a cuspon preserves its shape very well except for a phase shift. The reason for the numerical dispersion is thought due to the explicit method to solve the equation for the time evolution of mesh size. The detailed analysis is left for our another paper focusing on numerical solutions of the CH equation.

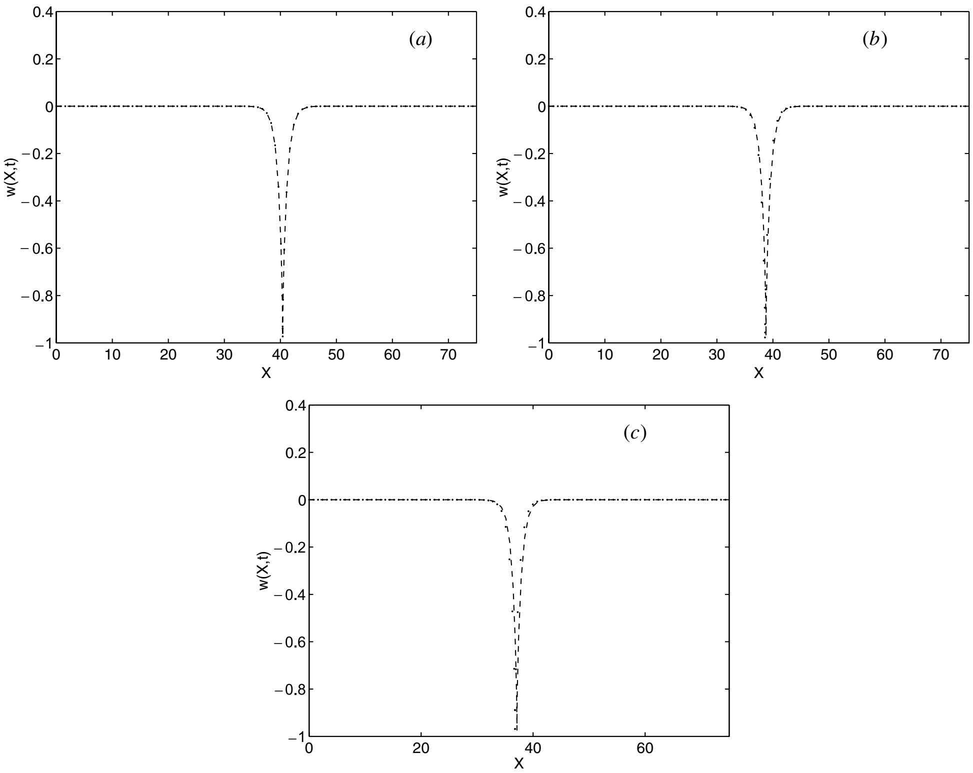

Figure 1. Numerical solution of one single-cuspon solution: (*a*) $t = 0.0$; (*b*) $t = 2.0$; (*c*) $t = 4.0$.

Example 2. 2-cuspon interaction. The parameters taken for the 2-cuspon solution are $p_1 = 11.0$, $p_2 = 10.5$, $c = 10.0$. Figure 2(*a*) shows the initial condition, and figures 2(*b*)–(*e*) display the process of collision at several different times. As far as we know, what is shown here is the first numerical demonstration for the cuspon–cuspon interaction due to the singularities of cuspon solutions. As shown in figure 2(*e*), the 2-cuspon solution regains its shape after the collision, only resulting in a phase shift. As mentioned in [18], the two cuspon points are always present during the collision.

Example 3. Soliton–cuspon interaction. Here we show two examples for the soliton–cuspon interaction with $c = 10.0$. In figure 3, we plot the interaction process between a soliton of $p_1 = 9.12$ and a cuspon of $p_2 = 10.98$ at several different times where the soliton and the cuspon have almost the same amplitude. It can be seen that another singularity point with infinite derivative (w_x) occurs when the collision starts ($t = 12.0$). As the collision goes on ($t = 14.4, 14.6, 14.8$), the soliton seems to 'eat up' the cuspon, and the profile looks like a complete elevation. However, the cuspon point is present at all times, especially, at $t = 14.6$, the profile becomes one symmetrical hump with a cuspon point in the middle of the hump.

In figure 4, we present another example of a collision between a soliton ($p_1 = 9.12$) and a cuspon ($p_2 = 10.5$), where the cuspon has a larger amplitude (2.0) than the soliton (1.0). Again, when the collision starts, another singularity point appears. As the collision goes on, the soliton is gradually absorbed by the cuspon. At $t = 10.3$, the whole profile looks like a single cuspon when the soliton is completely absorbed. Later on, the soliton emerges from the

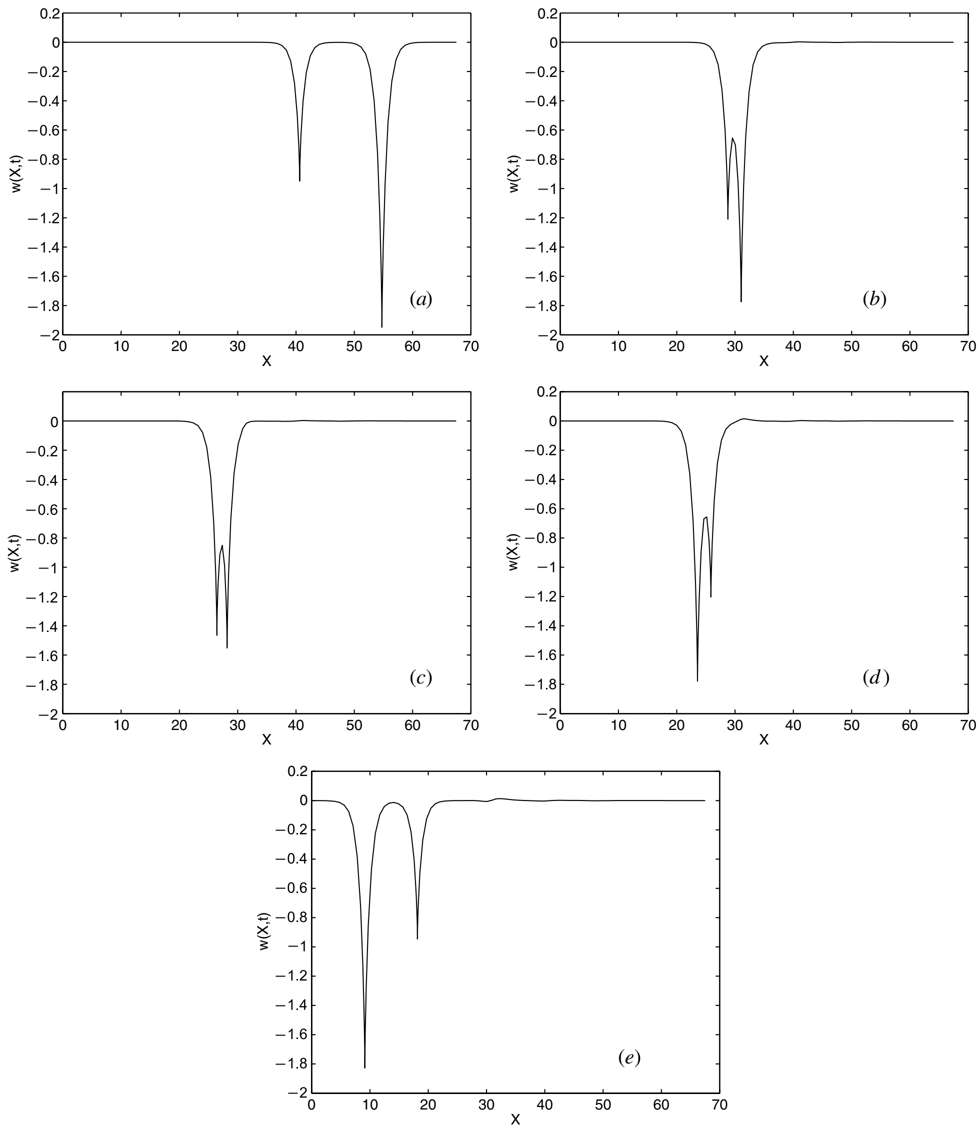

Figure 2. Numerical solution for the collision of the 2-cuspon solution with $p_1 = 11.0$, $p_2 = 10.5$, $c = 10.0$: (*a*) $t = 0.0$; (*b*) $t = 13.0$; (*c*) $t = 14.8$; (*d*) $t = 16.6$; (*e*) $t = 25.0$.

right until $t = 16$, the soliton and the cuspon recover their original shapes except for a phase shift when the collision is complete.

Example 4. Initial condition of non-exact soliton solutions. We show that the integrable scheme can also be applied for the initial value problem starting with a non-exact solution. We choose an initial condition in the following procedure. The mesh size is determined by

$$\delta_k = 2ac(1 - 0.8\text{sech}(2ka - W_x/3)), \tag{5.3}$$

where W_x (=8) is the width of computation, $N = 201$ is the number of grid in x-domain, $k = 1, \ldots, N-1$ and $a = W_x/(N-1) = 0.04$. Then the initial profile can be calculated through the second equation of the semi-discretization (3.29). The initial profile is plotted in

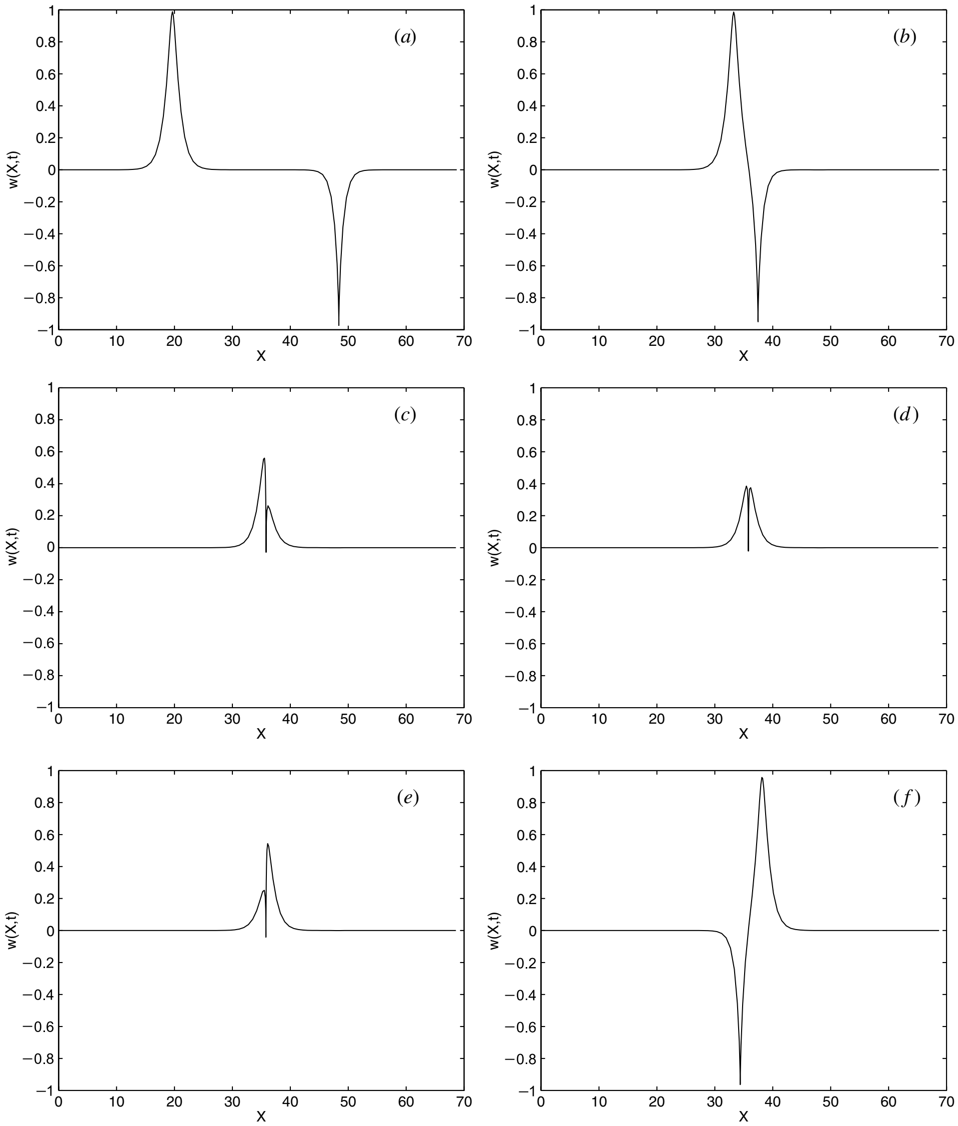

Figure 3. Numerical solution for the soliton–cuspon collision with $p_1 = 9.12$, $p_2 = 10.98$ and $c = 10.0$: (*a*) $t = 0.0$; (*b*) $t = 12.0$; (*c*) $t = 14.4$; (*d*) $t = 14.6$; (*e*) $t = 14.8$; (*f*) $t = 17.0$; (*g*) $t = 25.0$.

figure 5(*a*). Figures 5(*b*)–(*d*) show the evolution at $t = 10$, 20, 30, respectively. It can be seen that a soliton with large amplitude is developed first, and moving fast to the right. By $t = 30$, a second soliton with small amplitude is developed and a third soliton is born from the second soliton.

6. Concluding remarks

An integrable semi-discretization of the CH equation has been presented in this paper. Determinant formulas of the *N*-soliton solutions of both the continuous and semi-discrete CH

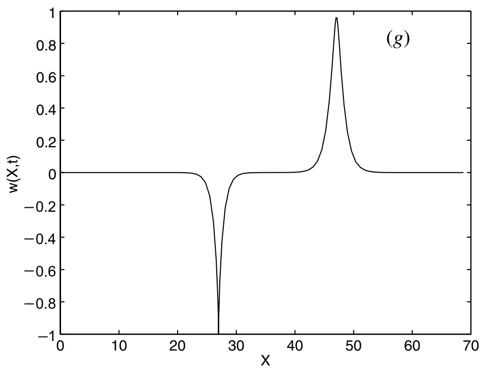

Figure 3. (Continued.)

equations have been derived. Multi-soliton, multi-cuspon and multi-soliton–cuspon solutions can be generated from the above determinant formulas. As further topics, we attempt to construct integrable semi-discretizations of the Degasperis–Procesi equation and other soliton equations possessing non-smooth solutions such as peakon, cuspon or loop-soliton solutions. We will address these issues in forthcoming papers.

Applying integrable discretizations of soliton equations to numerical computations remains a promising but not thoroughly explored subject. In this paper, even for relatively large mesh sizes, very accurate numerical solutions of cuspon–cuspon and soliton–cuspon interactions for the CH equations are achieved through our proposed integrable semi-discrete scheme. In addition, a numerical computation starting with a non-exact initial condition is performed with a satisfactory result. It is worth noting that the integrable semi-discrete scheme of the CH equation is also a self-adaptive method, which is of great interest in the area of numerical partial differential equations. We intend to extend this new idea of self-adaptive method to other PDEs in the near future.

Appendix A

In this appendix, we prove the differential formulas for τ_n, (2.9)–(2.16). Let us introduce a simplified notation,

$$\left|\psi^{(n_1)}, \psi^{(n_2)}, \ldots, \psi^{(n_N)}\right| = \begin{vmatrix} \psi_1^{(n_1)} & \psi_1^{(n_2)} & \cdots & \psi_1^{(n_N)} \\ \psi_2^{(n_1)} & \psi_2^{(n_2)} & \cdots & \psi_2^{(n_N)} \\ \vdots & \vdots & & \vdots \\ \psi_N^{(n_1)} & \psi_N^{(n_2)} & \cdots & \psi_N^{(n_N)} \end{vmatrix}. \tag{A.1}$$

In this notation, we have $\tau_n = |\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-1)}|$, thus differentiating τ_n by x and using (2.3) we obtain

$$\begin{aligned}\partial_x \tau_n &= \sum_{j=0}^{N-1} \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \partial_x \psi^{(n+j)}, \ldots, \psi^{(n+N-1)}\right| \\ &= \sum_{j=0}^{N-1} \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+j+1)} + c\psi^{(n+j)}, \ldots, \psi^{(n+N-1)}\right|\end{aligned}$$

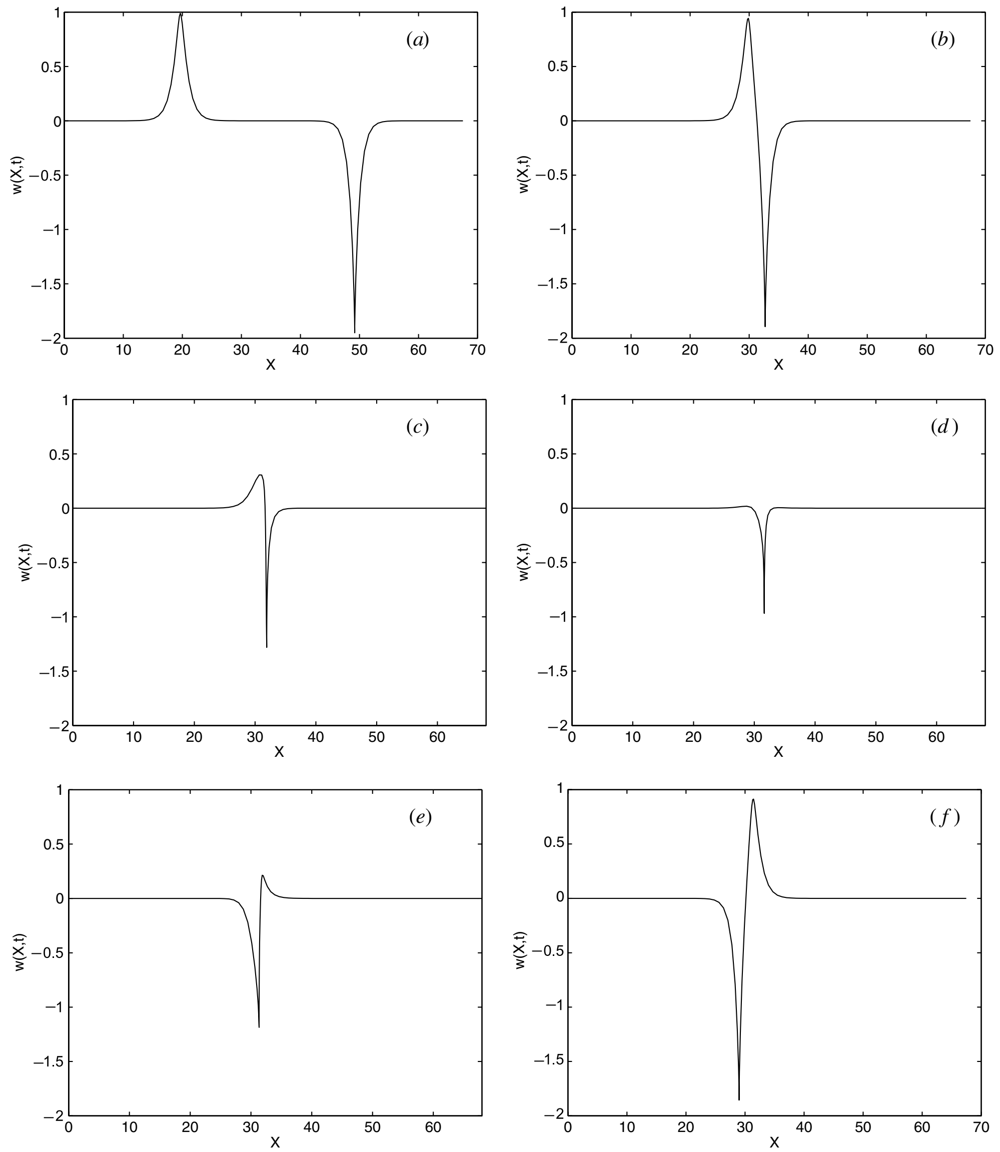

Figure 4. Numerical solution for the soliton–cuspon collision with $p_1 = 9.12$, $p_2 = 10.5$ and $c = 10.0$: (*a*) $t = 0.0$; (*b*) $t = 9.0$; (*c*) $t = 10.0$; (*d*) $t = 10.3$; (*e*) $t = 10.6$; (*f*) $t = 11.5$; (*g*) $t = 16.0$.

$$
\begin{aligned}
&= \sum_{j=0}^{N-1} \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+j+1)}, \ldots, \psi^{(n+N-1)}\right| \\
&\quad + \sum_{j=0}^{N-1} \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, c\psi^{(n+j)}, \ldots, \psi^{(n+N-1)}\right| \\
&= \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-2)}, \psi^{(n+N)}\right| + Nc|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-1)}|,
\end{aligned}
$$

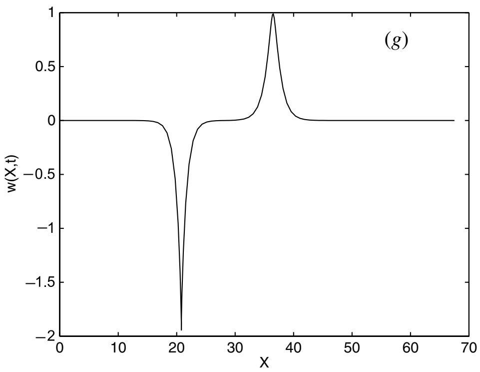

Figure 4. (Continued.)

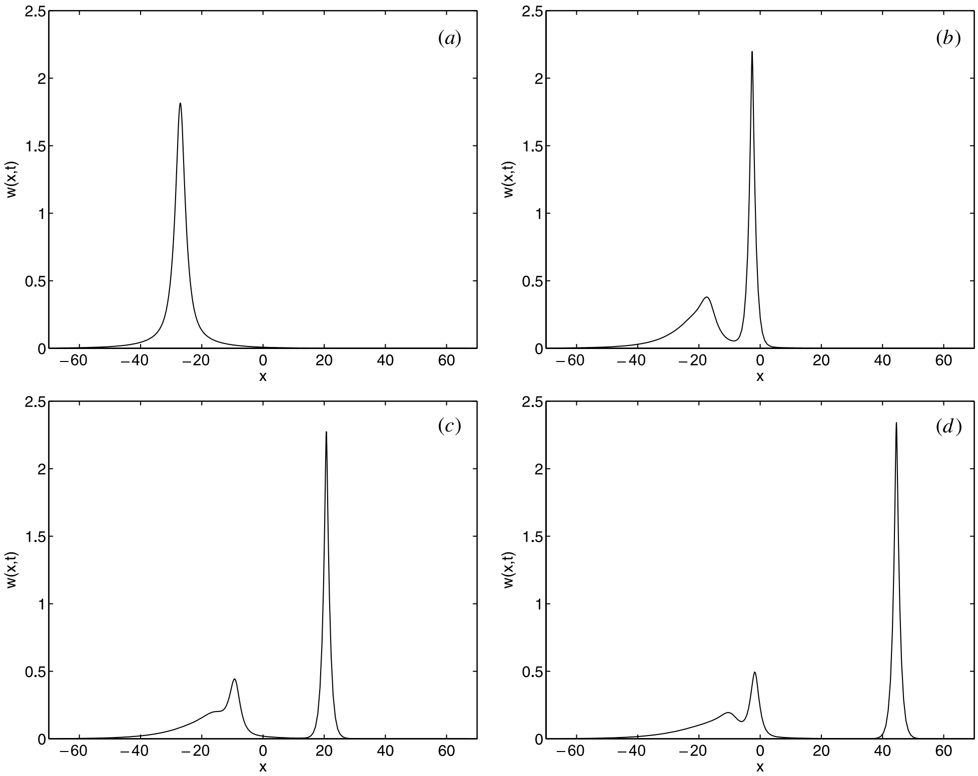

Figure 5. Numerical solution starting from an initial condition: (*a*) $t = 0.0$; (*b*) $t = 10.0$; (*c*) $t = 20.0$; (*d*) $t = 30.0$.

which gives (2.9). Similarly differentiating τ_n by y and using (2.4) we get

$$\partial_y \tau_n = \sum_{j=0}^{N-1} \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \partial_y \psi^{(n+j)}, \ldots, \psi^{(n+N-1)}\right|$$
$$= \sum_{j=0}^{N-1} \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+j+2)} + 2c\psi^{(n+j+1)} + c^2\psi^{(n+j)}, \ldots, \psi^{(n+N-1)}\right|$$

$$= \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-2)}, \psi^{(n+N+1)}\right|$$
$$- \left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-3)}, \psi^{(n+N-1)}, \psi^{(n+N)}\right|$$
$$+ 2c\left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-2)}, \psi^{(n+N)}\right| + Nc^2\left|\psi^{(n)}, \psi^{(n+1)}, \ldots, \psi^{(n+N-1)}\right|.$$

We obtain (2.10) from the above equation and (2.9). Equations (2.11) and (2.12) can be proved by using (2.5) and (2.6) in a similar way. Finally, equations (2.13)–(2.16) can be verified by differentiating (2.9) and (2.10) by t and s through similar calculations.

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